Intelligent states of the quantized radiation field associated with the Holstein-Primakoff realization of $\mathfrak{su}(2)$

2000 J. Phys. A: Math. Gen. 33 3347

(http://iopscience.iop.org/0305-4470/33/16/321)

J. Phys. A: Math. Gen. **33** (2000) 3347–3360. Printed in the UK PII: S0305-4470(00)11003-0

Intelligent states of the quantized radiation field associated with the Holstein–Primakoff realization of $\mathfrak{su}(2)$

Nai-Le Liu†‡, Zhi-Hu Sun‡ and Liu-Sheng Huang§

† CCAST(World Laboratory), PO Box 8730, Beijing, 100080, People's Republic of China
‡ Department of Material Science and Engineering, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China‖
§ National High Performance Computing Centre at Hefei, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China

E-mail: liunai@mail.ustc.edu.cn

Received 13 January 2000

Abstract. We study the intelligent states associated with the Holstein–Primakoff realization of $\mathfrak{su}(2)$. The explicit expressions of these states in terms of the Gauss hypergeometric functions are derived and their statistical properties are investigated in detail. It is shown that in some special or asymptotic cases these states turn out to be such important states as the binomial states, number states, Glauber coherent states, squeezed coherent states, etc in quantum optics.

1. Introduction

Other than the harmonic oscillator, whose dynamical symmetry group is the well-known Heisenberg–Weyl group H_4, the concept of coherent states (CS) [1, 2] has been generalized to describe systems associated with an arbitrary Lie group. Three different approaches have been developed to this problem [2]. For the harmonic oscillator case, all these approaches equivalently result in the Glauber CS [3], but for other Lie groups they lead to distinct quantum states. The Perelomov CS [4, 5], which are constructed by the action of group elements on a reference state of a group representation Hilbert space, have various properties (e.g. overcompleteness and invariance under the action of group representation operators) similar to the Glauber CS. In particular, taking the vacuum state as the reference state, one can identify the customary one- or two-mode squeezed states as Perelomov CS for appropriate boson realizations of $\mathfrak{su}(1,1)$ Lie algebra [6, 7]. Spin CS [8], which are similar to what are sometimes referred to in the literature as the atomic CS or the Bloch CS [9], are good $SU(2)$ examples. Two-mode $SU(2)$ CS using Schwinger's boson realization have also been discussed [10]. It is interesting to mention here that the binomial states [11] can be viewed as Perelomov CS connected to the Holstein–Primakoff realization (HPR) of $\mathfrak{su}(2)$ [12, 13], while the negative binomial states [14, 15] can be viewed as connected to the HPR of $\mathfrak{su}(1,1)$ [16, 17]. The multiboson CS based on the HPR of $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$ can be constructed by using generalized Bose operators [18].

In the second approach developed by Barut and Girardello [19] one deals with eigenstates of the lowering operator element of the Lie algebra. For the appropriate realizations of $\mathfrak{su}(1,1)$

‖ Address for correspondence.

these turn out to be Schrödinger cat states [20] in the one-mode case and pair CS [21] in the two-mode case. However, CS cannot be defined for compact Lie groups, e.g. the $SU(2)$ group, in this way.

There is a third approach to define the CS for various Lie groups. We are referring to constructing states leading to an equality in the Heisenberg uncertainty relation for Hermitian generators of a Lie group. The states defined in this way are called *intelligent states* (IS)—Aragone *et al* [22] were the first to use this terminology in the literature, when they derived the spin IS and pointed out the difference between minimum uncertainty states and IS. It goes without saying that the states providing an equality in the uncertainty relation do not, in general, reach a minimum uncertainty. In the last two decades, there has been much concern about IS, mainly in the context of quantum optics. One of the principal reasons for this concern lies in the close relationship between IS and squeezing [23]. In addition, the IS often show a variety of other nonclassical properties, such as antibunching effect, sub-Poissonian photon statistics [24], oscillatory photon counting distributions [25] and, in the two-mode case, violations of the Cauchy–Schwarz inequality [26]. Of particular interest are the IS with respect to single-mode two-photon realization of $\mathfrak{su}(1,1)$, which are in connection with the concept of amplitude-squared squeezing. These states were studied in great detail by Hillery and co-workers [27, 28], Prakash and Agarwal [29] and Marian [30]. In the corresponding two-mode case, the sum squeezing is identified with $SU(1,1)$ squeezing and the difference squeezing with $SU(2)$ squeezing [31]. The two-mode $SU(1,1)$ IS were studied by Gerry and Grobe [32]. Two-mode $SU(1,1)$ and $SU(2)$ IS have been demonstrated to be useful for improving the precision of measurements in quantum optics [33, 34]. Finally, it is of interest to refer to [35], where the IS associated with the HPR of $\mathfrak{su}(1,1)$ have been introduced and their properties investigated.

In this paper we study the IS associated with the HPR of $\mathfrak{su}(2)$. These do not appear to have been considered so far. We shall derive the explicit expressions of these states and discuss their interesting special cases and asymptotic behaviour. We shall also demonstrate their remarkable nonclassical properties. Let a ($a^\dagger$) denote the annihilation (creation) operator of a photon of a single-mode electromagnetic field ($[a, a^\dagger] = 1$) and N represent the number operator $a^\dagger a$. We consider the following set of operators:

$$J_3 = N - M/2 \qquad J_+ = a^\dagger\sqrt{M-N} \qquad J_- = J_+^\dagger = \sqrt{M-N}a \tag{1}$$

where M is a positive integer. These operators satisfy the commutation relations

$$[J_3, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_3 \tag{2}$$

which constitute the HPR of $\mathfrak{su}(2)$ [12]. The basis states of the relevant $(M+1)$-dimensional representation of $SU(2)$ coincide with the number (Fock) states

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \qquad n = 0, 1, \ldots, M \tag{3}$$

with J_3's eigenvalues being $-M/2, -M/2+1, \ldots, M/2$. If we alternatively introduce another pair of operators

$$J_1 = \tfrac{1}{2}(J_+ + J_-) \qquad J_2 = \frac{1}{2\mathrm{i}}(J_+ - J_-) \tag{4}$$

then one can see that they satisfy the commutation relation

$$[J_1, J_2] = \mathrm{i}J_3 \tag{5}$$

from which follows the uncertainty relation

$$(\Delta J_1)^2(\Delta J_2)^2 \geqslant \tfrac{1}{4}|\langle J_3\rangle|^2 \tag{6}$$

with $(\Delta J_i)^2 = \langle J_i^2\rangle - \langle J_i\rangle^2$ $(i = 1, 2)$. In view of (6) we shall call a state $SU(2)$ *squeezed* [23] if

$$(\Delta J_1)^2 < \tfrac{1}{2}|\langle J_3\rangle| \qquad \text{or} \qquad (\Delta J_2)^2 < \tfrac{1}{2}|\langle J_3\rangle|. \tag{7}$$

Let us now introduce the IS $|\psi\rangle$. They are defined to be those states which satisfy (6) as an equality. It is well known that such states must satisfy the following eigenvalue equation:

$$(J_1 + \mathrm{i}\lambda J_2)|\psi\rangle = \beta|\psi\rangle \tag{8}$$

where β is a complex number and λ is a real number. One can easily verify that ΔJ_i^2 are given by

$$(\Delta J_1)^2 = \frac{\lambda}{2}\langle\psi|J_3|\psi\rangle \qquad (\Delta J_2)^2 = \frac{1}{2\lambda}\langle\psi|J_3|\psi\rangle \tag{9}$$

from which it stands to reason that λ is an $SU(2)$ squeezing parameter: if $|\lambda| < 1$ (> 1), ΔJ_1 (ΔJ_2) is squeezed.

In section 2 we derive the analytic solutions to the eigenvalue equation (8). These solutions $|\psi\rangle$ are explicitly expressed in terms of the Gauss hypergeometric functions. Various interesting special cases and asymptotic properties of $|\psi\rangle$ are presented in section 3. Section 4 is devoted to demonstration of the nonclassical properties possessed by $|\psi\rangle$, such as the oscillatory behaviour of photon counting distributions and the antibunching effect as well as quadrature squeezing. We analyse the dependence of these effects on the parameters involved in $|\psi\rangle$ by making numerical evaluations. The Wigner function will also be investigated, for it helps provide insight into the nonclassical nature of the radiation field in $|\psi\rangle$. Finally, in section 5, we summarize the results and comment on the possibility of the generation of $|\psi\rangle$.

2. Analytic solutions

Let us rewrite the eigenvalue equation (8) in terms of J_+ and J_- as

$$\left(\frac{1+\lambda}{2}J_+ + \frac{1-\lambda}{2}J_-\right)|\psi\rangle = \beta|\psi\rangle. \tag{10}$$

When $\lambda = \pm 1$, (8) becomes $J_\pm|\psi\rangle = \beta|\psi\rangle$. The only eigenstate of J_+ (J_-) is the Fock state $|M\rangle$ (the vacuum state $|0\rangle$) with the corresponding eigenvalue 0. When $\lambda \neq \pm 1$, we follow the method Hillery and co-workers used in the investigation of IS for amplitude-squared squeezing [28] and introduce new states $|\varphi\rangle$ by

$$|\psi\rangle = S(\xi)|\varphi\rangle \qquad S(\xi) = \exp(\xi J_+ - \xi^* J_-) \tag{11}$$

where the parameter $\xi \equiv r\mathrm{e}^{\mathrm{i}\theta}$ will be determined later. Using the identities

$$S^{-1}(\xi)J_\pm S(\xi) = J_\pm \cos^2 r - J_3\mathrm{e}^{\mp\mathrm{i}\theta}\sin 2r - J_\mp \mathrm{e}^{\mp 2\mathrm{i}\theta}\sin^2 r \tag{12}$$

we obtain the equation satisfied by $|\varphi\rangle$:

$$\tfrac{1}{2}\{J_+[(1+\lambda)\cos^2 r - (1-\lambda)\mathrm{e}^{2\mathrm{i}\theta}\sin^2 r] + J_-[(1-\lambda)\cos^2 r - (1+\lambda)\mathrm{e}^{-2\mathrm{i}\theta}\sin^2 r]$$
$$- J_3[(1+\lambda)\mathrm{e}^{-\mathrm{i}\theta} + (1-\lambda)\mathrm{e}^{\mathrm{i}\theta}]\sin 2r\}|\varphi\rangle = \beta|\varphi\rangle. \tag{13}$$

Demanding that the coefficient of J_+ vanish leads to

$$\tan^2 r = \frac{1+\lambda}{1-\lambda}\mathrm{e}^{-2\mathrm{i}\theta}. \tag{14}$$

This suggests that we choose $\theta = 0$, $r = \arctan\sqrt{\frac{1+\lambda}{1-\lambda}}$ for $|\lambda| < 1$, and $\theta = \pi/2$, $r = \arctan\sqrt{\frac{\lambda+1}{\lambda-1}}$ for $|\lambda| > 1$. With these choices equation (13) becomes

$$\left(-\lambda J_- - \sqrt{1-\lambda^2}J_3\right)|\varphi\rangle = \beta|\varphi\rangle \qquad |\lambda| < 1 \tag{15}$$

$$\left(J_- + \mathrm{i}\frac{\lambda}{|\lambda|}\sqrt{\lambda^2-1}J_3\right)|\varphi\rangle = \beta|\varphi\rangle \qquad |\lambda| > 1. \tag{16}$$

We now concentrate only on deducing the exact solutions for the case $|\lambda| < 1$, the procedure for $|\lambda| > 1$ being quite similar. Expanding $|\varphi\rangle$ in terms of Fock states as $|\varphi\rangle = \sum_{m=0}^{M} c_m|m\rangle$ and substituting it into (15), and taking into account the well-known relations for the action of boson operators onto the Fock states (e.g. [3]), we obtain the following equations:

$$\left[\beta + \frac{M}{2}\sqrt{1-\lambda^2}\right]c_M = 0 \tag{17}$$

$$-\lambda c_{m+1}\sqrt{m+1}\sqrt{M-m} = \left[\beta + \sqrt{1-\lambda^2}(m-M/2)\right]c_m \qquad (m = 0, 1, \ldots, (M-1)). \tag{18}$$

From equation (17) we have $\beta = -\frac{M}{2}\sqrt{1-\lambda^2}$ or $c_M = 0$. If $\beta = -\frac{M}{2}\sqrt{1-\lambda^2}$, then we can obtain all the expansion coefficients c_m from (18). If, on the other hand, $c_M = 0$, then from (18) we have $[\beta + \sqrt{1-\lambda^2}(-1+M/2)]c_{M-1} = 0$ and there are still two possibilities: $\beta = \sqrt{1-\lambda^2}(-M/2+1)$ or $c_{M-1} = 0$. In general, $c_M \neq 0$ or for $K = 0, 1, \ldots, M-1$, $c_M = \cdots = c_{K+1} = 0$ but $c_K \neq 0$. Thus there exist $(M+1)$ eigenvalues which are given by

$$\beta = \sqrt{1-\lambda^2}(M/2-K) \qquad (K = 0, 1, \ldots, M). \tag{19}$$

Note that in the case $|\lambda| < 1$ all the eigenvalues are real. Inserting (19) into (18) we get the recursion relation

$$-\lambda c_{m+1}\sqrt{m+1}\sqrt{M-m} = \sqrt{1-\lambda^2}(m-K)c_m \qquad (K = 0, 1, \ldots, M; m = 0, 1, \ldots, K). \tag{20}$$

It then follows that

$$c_m = c_0\binom{K}{m}\sqrt{\binom{M}{m}^{-1}}\left(\sqrt{1-\lambda^2}/\lambda\right)^m \tag{21}$$

with c_0 being the normalization constant,

$$c_0 = \left\{\sum_{m=0}^{K}\binom{K}{m}^2\binom{M}{m}^{-1}[(1-\lambda^2)/\lambda^2]^m\right\}^{-1/2}$$
$$= \{{}_2F_1[-K, -K; -M; -(1-\lambda^2)/\lambda^2]\}^{-1/2}. \tag{22}$$

Here ${}_2F_1(a, b; c; \zeta)$ is the Gauss hypergeometric function [36]. Then, inserting a complete set of number states into equation (11), we can express $|\psi\rangle$ for the case $|\lambda| < 1$ as

$$|\psi\rangle^{(|\lambda|<1)} = \sum_{n=0}^{M} d_n^{(|\lambda|<1)}(M, K, \lambda)|n\rangle \tag{23}$$

where

$$d_n^{(|\lambda|<1)}(M, K, \lambda) = \sum_{m=0}^{K} c_m\langle n|S\left(\arctan\sqrt{\frac{1+\lambda}{1-\lambda}}\right)|m\rangle. \tag{24}$$

To get the explicit expressions of $d_n^{(|\lambda|<1)}(M, K, \lambda)$ we have to evaluate the matrix element $\langle n|S(\arctan\sqrt{\frac{1+\lambda}{1-\lambda}})|m\rangle$. This can be achieved by using the Baker–Campbell–Hausdorff formula for $SU(2)$ [37]

$$\exp(\xi J_+ - \xi^* J_-) = \exp\left[\left(\frac{\xi}{|\xi|}\tan|\xi|\right)J_+\right]\exp[-2(\ln\cos|\xi|)J_3]\exp\left[-\left(\frac{\xi^*}{|\xi|}\tan|\xi|\right)J_-\right] \tag{25}$$

and the following two identities:

$$\left(\sqrt{M-N}a\right)^n|m\rangle = \begin{cases} n!\sqrt{\binom{m}{n}\binom{M-m+n}{n}}|m-n\rangle & n \leqslant m \\ 0 & n > m \end{cases} \tag{26}$$

$$\left(a^\dagger\sqrt{M-N}\right)^n|m\rangle = \begin{cases} n!\sqrt{\binom{M-m}{n}\binom{n+m}{n}}|m+n\rangle & n \leqslant M-m \\ 0 & n > M-m. \end{cases} \tag{27}$$

After some manipulation, we obtain the explicit expression of $\langle n|S(\arctan\sqrt{\frac{1+\lambda}{1-\lambda}})|m\rangle$ which reads

$$\begin{aligned}\langle n|S\left(\arctan\sqrt{\frac{1+\lambda}{1-\lambda}}\right)|m\rangle &= (-)^m\left(\sqrt{\frac{1+\lambda}{1-\lambda}}\right)^{m+n}\sqrt{\binom{M}{n}\binom{M}{m}} \\ &\quad\times\left(\sqrt{\frac{1-\lambda}{2}}\right)^M {}_2F_1[-n,-m;-M;2/(1+\lambda)].\end{aligned} \tag{28}$$

Inserting (21), (22) and (28) into (24) and making use of the formula [36]

$$\sum_{n=0}^{\infty}\binom{\eta}{n}t^n{}_2F_1(-n,b;c;\zeta) = (1+t)^\eta{}_2F_1[-\eta,b;c;t\zeta/(1+t)] \tag{29}$$

we find that $d_n^{(|\lambda|<1)}(M, K, \lambda)$ can be written in a closed form, i.e.

$$\begin{aligned}d_n^{(|\lambda|<1)}(M,K,\lambda) &= \{{}_2F_1[-K,-K;-M;-(1-\lambda^2)/\lambda^2]\}^{-1/2}\left(\sqrt{\frac{1-\lambda}{2}}\right)^M(-\lambda)^{-K} \\ &\quad\times\sqrt{\binom{M}{n}}\left(\sqrt{\frac{1+\lambda}{1-\lambda}}\right)^n{}_2F_1(-n,-K;-M;2).\end{aligned} \tag{30}$$

Proceeding in the same way, we obtain the explicit expressions of normalized IS $|\psi\rangle$ for the case $|\lambda| > 1$

$$|\psi\rangle^{(|\lambda|>1)} = \sum_{n=0}^{M} d_n^{(|\lambda|>1)}(M,K,\lambda)|n\rangle \tag{31}$$

where

$$\begin{aligned}d_n^{(|\lambda|>1)}(M,K,\lambda) &= \{{}_2F_1[-K,-K;-M;-(\lambda^2-1)]\}^{-1/2}\left(\sqrt{\frac{\lambda-1}{2\lambda}}\right)^M(-\lambda)^K \\ &\quad\times\sqrt{\binom{M}{n}}\left(\mathrm{i}\sqrt{\frac{\lambda+1}{\lambda-1}}\right)^n{}_2F_1(-n,-K;-M;2).\end{aligned} \tag{32}$$

The corresponding eigenvalues are

$$\beta = \mathrm{i}\frac{\lambda}{|\lambda|}\sqrt{\lambda^2 - 1}(K - M/2) \qquad (K = 0, 1, \ldots, M). \tag{33}$$

All the eigenvalues in (33) are, contrary to the case of $|\lambda| < 1$, purely imaginary. We recall that the eigenvalues corresponding to the $SU(1,1)$ IS are, in general, arbitrary complex numbers (e.g. [28–30]); however, here we see that for the $SU(2)$ IS the eigenvalues must be *either real or purely imaginary*. This seems to be an underlying distinction between $\mathfrak{su}(2)$ and $\mathfrak{su}(1,1)$. So far, we have completed the derivation of the analytic solutions. These explicit expressions enable us to directly discuss the various properties of the IS. Before concluding this section let us show that the photon-counting distribution of $|\psi\rangle$ is *symmetric* with respect to the parameters λ and K, respectively. From (30) and (32) it is easily seen that

$$d_n^{(|\lambda|>1)}(M, K, \lambda = \lambda_0) = \mathrm{e}^{\mathrm{i}n\pi/2} d_n^{(|\lambda|<1)}(M, K, \lambda = 1/\lambda_0) \tag{34}$$

where λ_0 is an arbitrary real number whose absolute value is greater than 1, thereby implying that the distribution for $\lambda = \lambda_0$ (λ_0 is an arbitrary real number) is entirely the same as the distribution for $\lambda = 1/\lambda_0$. In addition, with the use of the transformations [36]

$$_2F_1(a, b; c; \zeta) = (1 - \zeta)^{-a} {}_2F_1[a, c - b; c; \zeta/(\zeta - 1)] \tag{35}$$

$$= (1 - \zeta)^{c-a-b} {}_2F_1(c - a, c - b; c; \zeta) \tag{36}$$

quite simply one can verify

$$d_n(M, M - K, \lambda) = (-)^{M+n} d_n(M, K, \lambda) \tag{37}$$

which means that the photon-counting distributions for $K = K_0$ (K_0 is an arbitrary non-negative integer less than or equal to M) and $K = M - K_0$ are exactly the same.

3. Special cases and asymptotic behaviour of $|\psi\rangle$

3.1. Special cases

In 1985, Stoler *et al* [11] introduced the binomial states (BS) of a quantized radiation field:

$$|M, p, \theta\rangle = \sum_{n=0}^{M} \sqrt{\binom{M}{n} p^n (1-p)^{M-n}} \mathrm{e}^{\mathrm{i}n\theta} |n\rangle \qquad 0 < p < 1. \tag{38}$$

Since then, these states have attracted attention, mainly due to the fact that they interpolate the number states and the Glauber CS [38]. For this reason the BS are termed a class of *intermediate states* in quantum optics. As pointed out by Fan and Jing [13], the BS can be viewed as Perelomov CS connected with the HPR of $\mathfrak{su}(2)$, namely

$$|M, p, \theta\rangle = \exp[r\mathrm{e}^{\mathrm{i}\theta}(J_+ - J_-)]|0\rangle \qquad r = \arcsin\sqrt{p}. \tag{39}$$

Using the formulae $_2F_1[0, b; c; \zeta] = 1$ and $_2F_1[-a, -b; -b; -\zeta] = (1+\zeta)^a$ in (30) and (32) we find that $|\psi\rangle$ reduce to special BS when $K = 0$ or $K = M$:

$$|\psi\rangle^{(|\lambda|<1)} = \begin{cases} |M, (1+\lambda)/2, 0\rangle & K = 0 \\ |M, (1+\lambda)/2, \pi\rangle & K = M \end{cases} \tag{40}$$

$$|\psi\rangle^{(|\lambda|>1)} = \begin{cases} |M, (1+\lambda)/(2\lambda), \pi/2\rangle & K = 0 \\ |M, (1+\lambda)/(2\lambda), -\pi/2\rangle & K = M. \end{cases} \tag{41}$$

Further, in two limiting cases from BS, of course, $|\psi\rangle$ respectively degenerate to the number states and the Glauber CS.

We now consider another kind of special case: M is an even number and $K = M/2$. This corresponds to eigenvalues β being zero. The use of the identity [36]

$$(c-a)_2F_1(a-1,b;c;\zeta) + (2a-c-a\zeta+b\zeta)_2F_1(a,b;c;\zeta) \\ +a(\zeta-1)_2F_1(a+1,b;c;\zeta) = 0 \tag{42}$$

enables us to obtain

$$(n-2K)_2F_1(-n-1,-K;-2K;2) = n_2F_1(-n+1,-K;-2K;2). \tag{43}$$

Then, taking into account ${}_2F_1(-1,-K;-2K;2) = 0$ and ${}_2F_1(0,-K;-2K;2) = 1$, we derive

$${}_2F_1(-n,-K;-2K;2) = \begin{cases} (-)^{n/2}\dfrac{(n-1)!!(2K-n-1)!!}{(2K-1)!!} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd.} \end{cases} \tag{44}$$

Inserting (44) into (30) and (32) we find that all odd coefficients vanish and only states with even photon numbers survive. The resulting states read

$$|\psi\rangle = d_0 \sum_{n=0}^{K} \left[\left(-\frac{1+\lambda}{1-\lambda}\right)^n \sqrt{\frac{(2K)!!(2K-2n-1)!!}{(2K-1)!!(2K-2n)!!}} \frac{\sqrt{(2n)!}}{2^n n!} \right] |2n\rangle \tag{45}$$

where

$$d_0 = \begin{cases} \{{}_2F_1[-K,-K;-2K;-(1-\lambda^2)/\lambda^2]\}^{-1/2}[(1-\lambda)/(-2\lambda)]^K & |\lambda| < 1 \\ \{{}_2F_1[-K,-K;-2K;-(\lambda^2-1)]\}^{-1/2}[-(\lambda-1)/2]^K & |\lambda| > 1. \end{cases} \tag{46}$$

We also note from equation (20) that when $\lambda = 0$, $(m-K)c_m = 0$, which means $c_m = 0$ for $m = 0, 1, \ldots, (K-1)$ but $c_K \neq 0$. Therefore, $|\varphi\rangle^{(\lambda=0)}$ is simply the Fock state $|K\rangle$ and

$$|\psi\rangle^{(\lambda=0)} = S(\pi/4)|K\rangle. \tag{47}$$

3.2. Asymptotic properties

3.2.1. Limit leading to the squeezed vacuum. Consider $\lambda < 0$, $K = [M/2]$, where the notation $[x]$ denotes the greatest integer less than or equal to x. In the limit $M \to \infty$, it is obvious from (19) and (33) that β keeps finite and thus the eigenvalue equation (10) degenerates to

$$(\mu a + \nu a^\dagger)|\psi\rangle = 0 \tag{48}$$

where

$$\mu = \frac{1-\lambda}{2\sqrt{-\lambda}} \qquad \upsilon = \frac{1+\lambda}{2\sqrt{-\lambda}} \qquad \mu^2 - \nu^2 = 1. \tag{49}$$

Equation (48) is nothing other than the eigenvalue equation satisfied by the squeezed vacuum [6].

3.2.2. Limits leading to squeezed CS and further to Glauber CS. Consider $\lambda < 0$, $K = [M/2] - \gamma[\sqrt{M}]$ with γ being a positive integral parameter. In the limit $M \to \infty$ but keeping γ finite (i.e. $\gamma \ll \sqrt{M}$), the eigenvalue equation (10) reduces to

$$(\mu a + \nu a^\dagger)|\psi\rangle = \tau|\psi\rangle \tag{50}$$

wherein μ and ν remain as given above and τ is given by

$$\tau = \begin{cases} \gamma\sqrt{\lambda - (1/\lambda)} & -1 < \lambda < 0 \\ \mathrm{i}\gamma\sqrt{-\lambda + (1/\lambda)} & \lambda < -1 \end{cases} \tag{51}$$

thereby implying that in this limit $|\psi\rangle$ tend to the squeezed CS (i.e. two-photon CS). In a further limit $\lambda \to (-1)^+$ (i.e. the right-hand limit of -1), $\gamma \to \infty$ (but keeping $\gamma \ll \sqrt{M}$, needless to say) with $\gamma\sqrt{\lambda+1} = \alpha/\sqrt{2}$ finite (α is a real number), the eigenvalue equation (10) reduces to

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{52}$$

$|\alpha\rangle$ being the Glauber CS. Similarly, in the limit $\lambda \to (-1)^-$ (i.e. the left-hand limit of -1), $\gamma \to \infty$, while $\gamma\sqrt{-(\lambda+1)} = \alpha/\sqrt{2}$ are finite, $|\psi\rangle$ approach the Glauber CS $|\mathrm{i}\alpha\rangle$. All the facts above indicate that the IS $|\psi\rangle$ provide a way of treating the relations amongst different important states in quantum optics.

4. Nonclassical properties

As remarked in section 2, there exist *parametric symmetries* with respect to λ and K in the photon-counting distribution. These lead to corresponding *parametric symmetries* in the properties (e.g., the sub-Poissonian distribution and the antibunching effect as well as the Wigner function) which are entirely determined by the photon-counting distribution. That is, when we examine one such property we should only consider the region $|\lambda| < 1$ and the values of K less than or equal to $M/2$. Nevertheless, one should note that the squeezing properties depend not only on the photon-counting distribution, but also on the phase of the expansion coefficients of $|\psi\rangle$. We realize from the phase factor $\mathrm{e}^{\mathrm{i}n\pi/2}$ in equation (34) the necessity of considering the whole range of λ (from $-\infty$ to $+\infty$) when studying the squeezing properties of $|\psi\rangle$.

In figure 1 we plot the photon-counting distribution $P_n^{(|\lambda|<1)} = |d_n^{(|\lambda|<1)}(M, K, \lambda)|^2$ for $M = 40$, $K = 10$ and different values of λ. The oscillations in the distribution can easily be seen. It is known that oscillatory behaviour of the photon-counting distribution is manifestly nonclassical and can be associated with interference in phase space [25]. The centre of the distribution is rather sensitive to λ. For given parameters M and K, the maximum of the distribution shifts to larger values of n with increasing λ. In the case $\lambda = 0$ (corresponding to the state $S(\pi/4)|K\rangle$), the distribution is symmetric with respect to n; this fact can also be inferred from taking $\lambda = 0$ in (30) and making use of (35). In figure 2 we plot $P_n^{(|\lambda|<1)}$ for $M = 80$, $K = 40$ and $\lambda = -0.1$, which clearly displays the absence of odd photon numbers and approximates to the distribution of a squeezed vacuum, as demonstrated before.

A field is antibunched if its second-order correlation function $g^{(2)}(0) < 1$ [39], namely

$$g^{(2)}(0) = \frac{\langle N^2\rangle - \langle N\rangle}{\langle N\rangle^2} < 1. \tag{53}$$

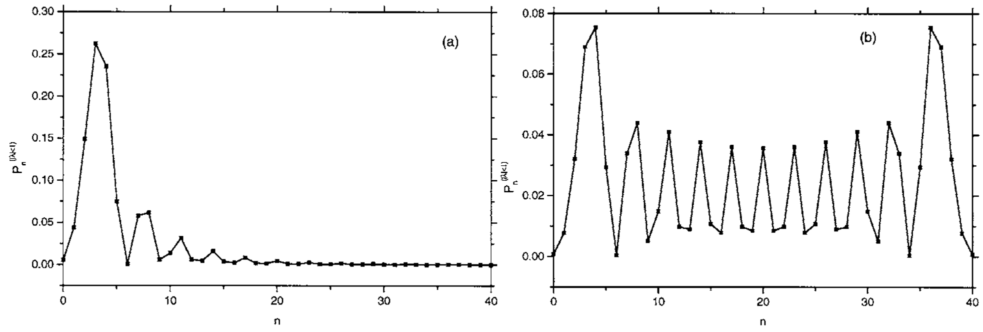
The sub-Poissonian character and antibunching effect are always coincident for single-mode and time-independent fields. In our numerical study of $g^{(2)}(0)$, we plot this function against the parameter λ (in the region $|\lambda| < 1$) for $M = 10$ and different values of K (see figure 3). When $K = 0$ (BS), $g^{(2)}(0)$ keeps 0.9 (i.e. $(1 - 1/M)$) as expected and so the antibunching behaviour persists for the whole interval of λ. When $K \neq 0$ it is observed that there are always some intervals of λ within which the IS exhibit antibunching effect.

The quadrature operators of the single-mode field are defined as

$$X = \frac{1}{\sqrt{2}}(a + a^\dagger) \qquad P = \frac{1}{\sqrt{2}\mathrm{i}}(a - a^\dagger). \tag{54}$$

They satisfy the commutation relation $[X, P] = i$ and consequently their variances $(\Delta X)^2 = \langle X^2\rangle - \langle X\rangle^2$, $(\Delta P)^2 = \langle P^2\rangle - \langle P\rangle^2$ obey the Heisenberg uncertainty relation

$$(\Delta X)^2(\Delta P)^2 \geqslant \tfrac{1}{4}. \tag{55}$$



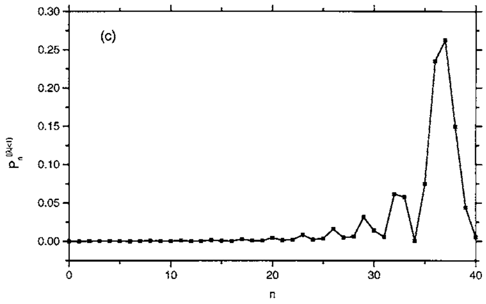


Figure 1. The photon-counting distributions $P_n^{(|\lambda|<1)}$ of the IS $|\psi\rangle$ for $M = 40$, $K = 10$ and different values of λ: (*a*) $\lambda = -0.1$, (*b*) $\lambda = 0$, (*c*) $\lambda = 0.1$.

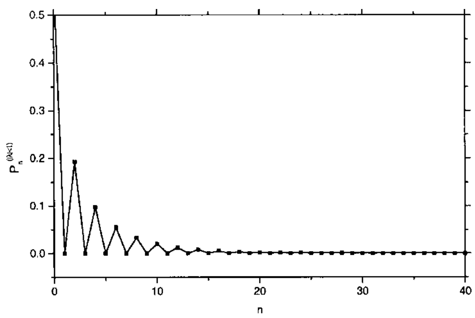


Figure 2. The photon-counting distribution $P_n^{(|\lambda|<1)}$ of the IS $|\psi\rangle$ for $M = 80$, $K = 40$ and $\lambda = -0.1$.

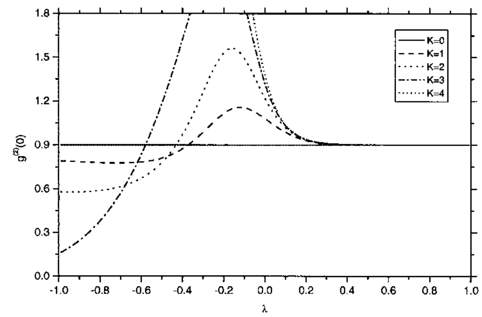


Figure 3. The second-order correlation function $g^{(2)}(0)$ of the IS $|\psi\rangle$ as a function of λ for $M = 10$ and different values of K.

The field is said to be squeezed in the X (P) quadrature if $(\Delta X)^2 < \frac{1}{2}$ ($(\Delta P)^2 < \frac{1}{2}$). For the sake of convenience, we define the squeezing indices as

$$S_x = 2(\Delta X)^2 - 1 \qquad S_p = 2(\Delta P)^2 - 1. \tag{56}$$

When $S_x < 0$ ($S_p < 0$), the field is squeezed in the X (P) quadrature. From equation (54) S_x and S_p are expressed as

$$S_x = \langle a^2 + a^{\dagger 2}\rangle + 2\langle N\rangle - \langle a + a^\dagger\rangle^2 \tag{57}$$

$$S_p = -\langle a^2 + a^{\dagger 2}\rangle + 2\langle N\rangle + \langle a - a^\dagger\rangle^2. \tag{58}$$

We have plotted S_x and S_p in figure 4, against the parameter λ for $M = 10$ and different values of K (as remarked before, we should only consider the values of K less than or equal to $M/2$). One can see that the IS $|\psi\rangle$ do exhibit squeezing in the quadrature X or P; however, the depth of squeezing and the range of λ over which squeezing is observed are very sensitive

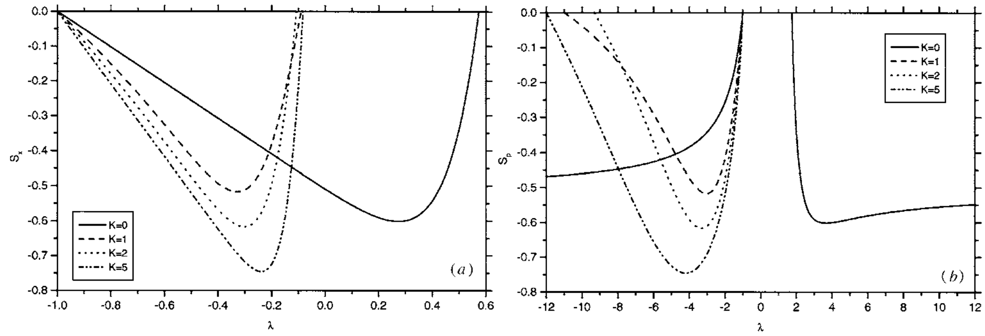


Figure 4. Squeezing indices S_x (a) and S_p (b) of the IS $|\psi\rangle$ as functions of λ for $M = 10$ and different values of K.

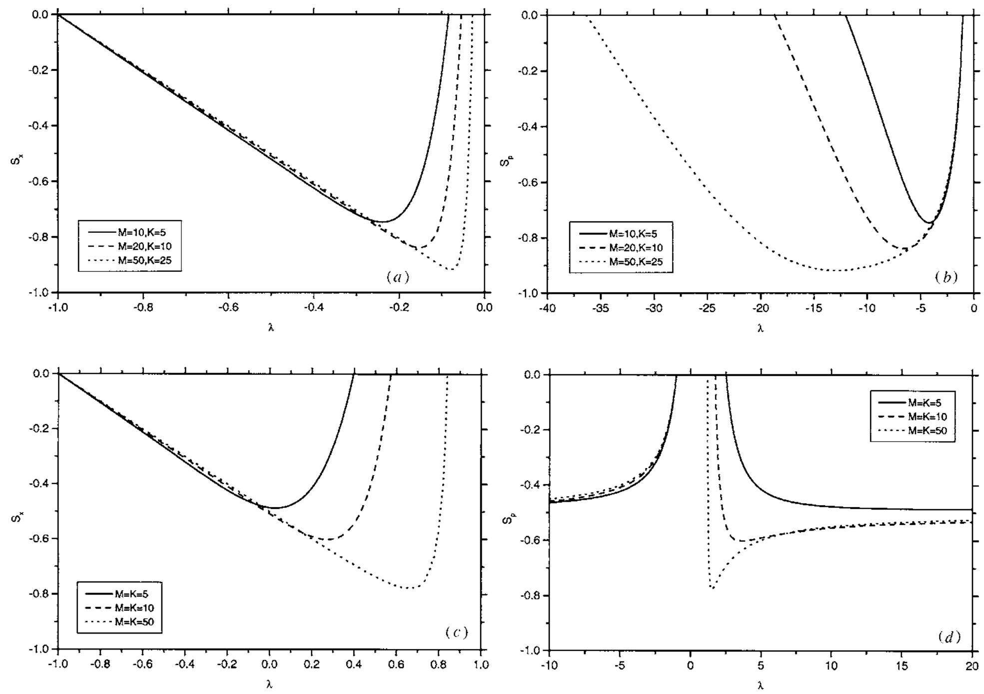


Figure 5. Squeezing indices S_x (a, c) and S_p (b, d) of the IS $|\psi\rangle$ as functions of λ for different values of M and K such that $M = K$ (c, d) or $M = 2K$ (a, b).

to the values of K. When $K = 0$ (corresponding to BS), X-squeezing is present in the range $-1 < \lambda < 0.57$, while the P-squeezing regions are within the intervals $|\lambda| > 1$. When $K \neq 0$, the X-squeezing region is within the interval $(-1, 0)$ while the P-squeezing region is within $(-\infty, -1)$; as K increases, both X-squeezing and P-squeezing become more effective. In order to study the dependence of squeezing on the maximum excitation number M, we have plotted S_x and S_p in figure 5, against the parameter λ for different values of M and K such that $M = K$ or $M = 2K$. It turns out that increasing M can enhance squeezing and broaden the squeezing range.

Quasi-probability distributions can help provide insight into the nonclassical nature of radiation fields. Of these, the Wigner function [40] plays an exceptional role as it contains complete information about the state of the system, i.e. it carries the same information as the density operator or the corresponding wavefunction. The Wigner function is defined as the Fourier transform of the characteristic function, associated with the symmetrical order of the annihilation and creation operators. Alternatively, the Wigner function for an arbitrary density operator ρ may be given by [41]

$$W(z) = \frac{2}{\pi}\,\mathrm{Tr}[\rho D(2z)\exp(\mathrm{i}\pi N)] \tag{59}$$

where $D(z) = \mathrm{e}^{za^\dagger - z^* a}$ is the displacement operator of the harmonic oscillator and z is a complex c-number. Inserting $\rho = |\psi\rangle\langle\psi|$ we obtain the Wigner function of the IS $|\psi\rangle$ as

$$W(z) = \frac{2}{\pi}\sum_{n=0}^{M}\sum_{m=0}^{M} d_n^*(M, K, \lambda) d_m(M, K, \lambda)(-1)^m \chi_{nm}(2z). \tag{60}$$

Here [42]

$$\chi_{nm}(z) = \langle n|D(z)|m\rangle = \begin{cases} \sqrt{\dfrac{m!}{n!}}\mathrm{e}^{-|z|^2/2} z^{n-m} L_m^{n-m}(|z|^2) & n \geqslant m \\ \sqrt{\dfrac{n!}{m!}}\mathrm{e}^{-|z|^2/2}(-z^*)^{m-n} L_n^{m-n}(|z|^2) & n < m \end{cases} \tag{61}$$

wherein $L_m^\nu(x) \equiv \sum_{l=0}^{m}\binom{m+\nu}{m-l}\frac{(-x)^l}{l!}$ is the associated Laguerre polynomial. From (60) it is apparent that $W(z)$ is a symmetric function in Im (z). As stated before, we should only consider the region $|\lambda| < 1$. We have studied numerically the behaviour of the Wigner function $W(z)$ as a function of $z = \mathrm{Re}\,(z) + \mathrm{iIm}\,(z)$ for $M = 2$, $K = 0, 1$ and different values of λ. The results are shown in the sequence of figures 6(*a*)–(*h*). With reference to figures 6(*a*) and (*b*) we see that when λ is close to -1, the function has an almost Gaussian shape centred in the origin, and is largely insensitive to change in K. This is reasonable because of the dominance of the effect of the vacuum state over the effects of the higher excitations. As λ increases ($\lambda = -0.1$, for instance), which means that the vacuum state begins to lose its higher probability in the number state expansion, some negative part of the distribution appears. As is well known, the negativity of the Wigner function signifies nonclassical effects. When the value of λ is even larger, the negative part is even larger, $W(z)$ deviates far away from the Gaussian distribution and the rings characteristic of a number state start being formed, as we can appreciate in the plot of $\lambda = 0.8$. When $\lambda = 1$, the number state is produced.

5. Concluding remarks

In this work we have introduced the IS associated with the HPR of $\mathfrak{su}(2)$ and shown that they display strong nonclassical properties such as the oscillatory behaviour of photon-counting distributions and the antibunching effect as well as quadrature squeezing. These states turn out to be a new class of *intermediate states*, for they take different important states as their special or limiting cases.

We finally briefly discuss the possibility for the realization of the IS $|\psi\rangle$. As a matter of fact, although the generation of pure superposition states has been a major subject in quantum optics, it does not seem to be a task of immediate implementation. To the author's knowledge, even the realization of the BS is not available yet. Recently, some progress has been made in schemes for realizing arbitrary pure states. In [43], for example, a method based upon a *nonunitary* 'collapse' of the state vector of the cavity-field mode via atom ground-state

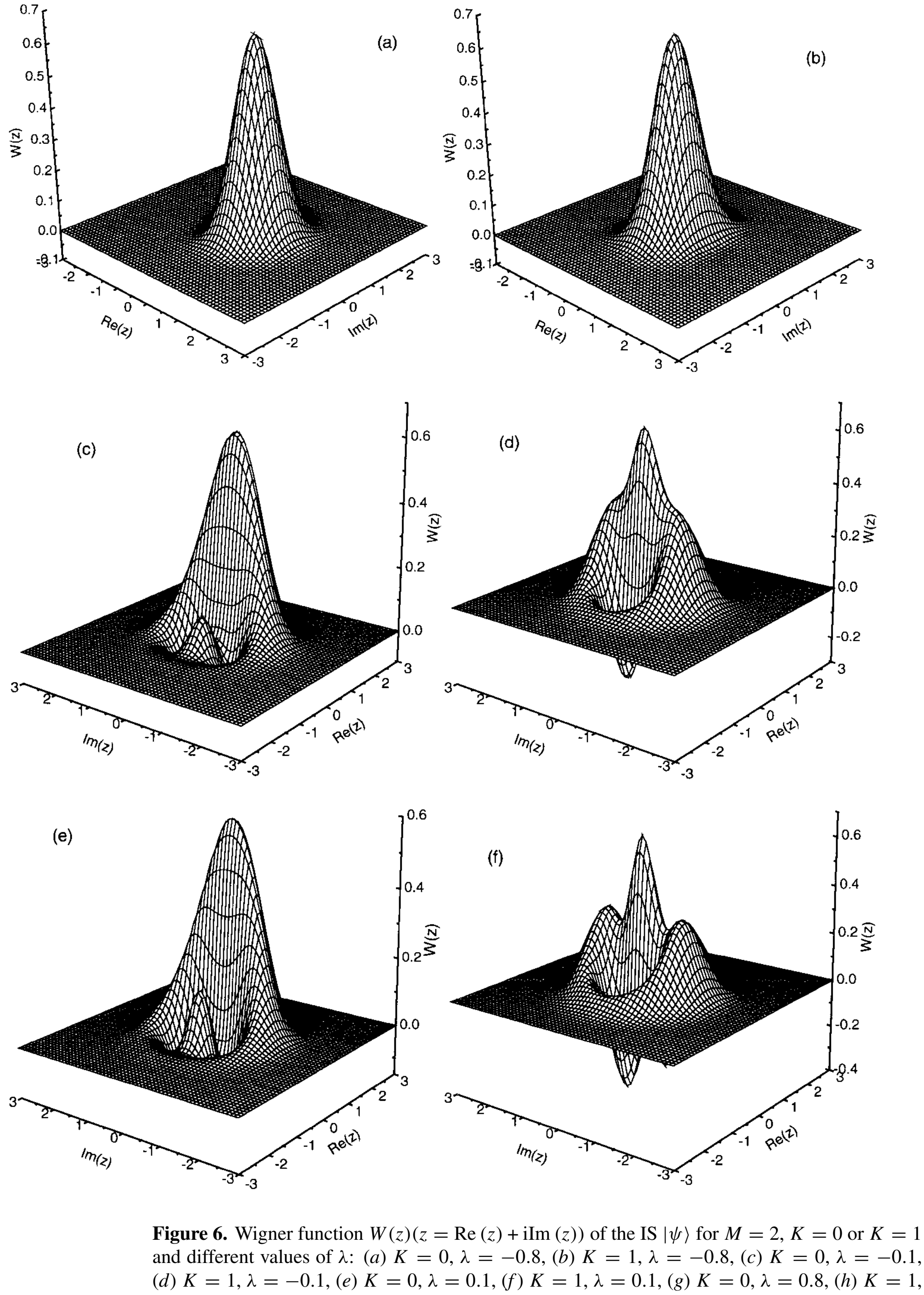


Figure 6. Wigner function $W(z)(z = \mathrm{Re}\,(z) + \mathrm{iIm}\,(z))$ of the IS $|\psi\rangle$ for $M = 2$, $K = 0$ or $K = 1$ and different values of λ: (*a*) $K = 0$, $\lambda = -0.8$, (*b*) $K = 1$, $\lambda = -0.8$, (*c*) $K = 0$, $\lambda = -0.1$, (*d*) $K = 1$, $\lambda = -0.1$, (*e*) $K = 0$, $\lambda = 0.1$, (*f*) $K = 1$, $\lambda = 0.1$, (*g*) $K = 0$, $\lambda = 0.8$, (*h*) $K = 1$, $\lambda = 0.8$. (Continued opposite.)

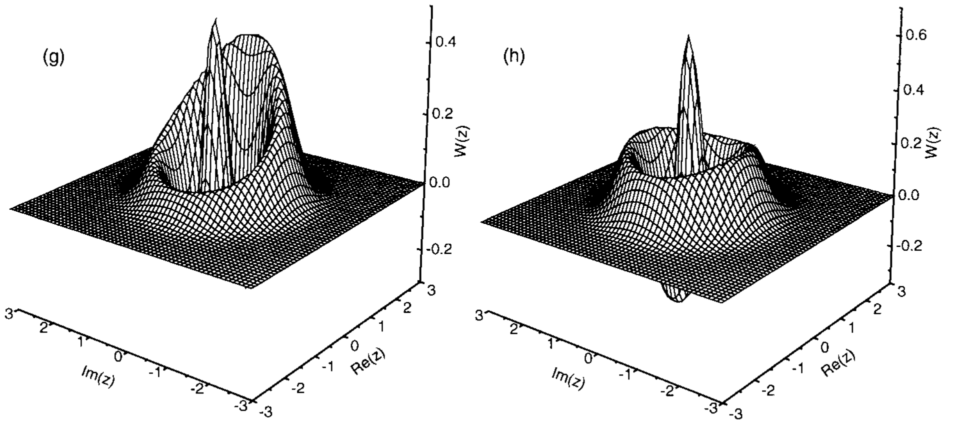


Figure 6. (Continued)

measurement is proposed for preparing a cavity-field mode undergoing a Jaynes–Cummings dynamics in any superposition of a finite number of Fock states in principle. The scheme in [44], however, uses a cavity QED *unitary* time-dependent interaction. With respect to these two methods, it has been argued by the authors of [45] that 'both approaches involve individual atoms interacting with a single-mode cavity field, which would demand extraordinary control in a generation experiment. It is therefore interesting to seek alternative methods for the generation of nonclassical light'. The method proposed in [45] is to construct a Hamiltonian which would allow the use of some kind of nonlinear interaction for the production of arbitrary pure states. In a more recent paper [46] it is shown that arbitrary pure quantum states can be realized by a succession of alternate state displacement and single-photon adding. Based on the above significant studies, the IS $|\psi\rangle$ will hopefully be produced in the not too distant future.

Acknowledgment

This work was supported by the National High Performance Computing Foundation of China.

References

[1] Klauder J R and Skagerstam B S 1985 *Coherent States: Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
[2] Zhang W-M, Feng D H and Gilmore R 1990 *Rev. Mod. Phys.* **62** 867
[3] Glauber R J 1963 *Phys. Rev.* **131** 2766
[4] Perelomov A M 1972 *Commun. Math. Phys.* **26** 222
[5] Perelomov A M 1986 *Generalized Coherent States and their Applications* (Berlin: Springer)
[6] Yuen H P 1976 *Phys. Rev.* A **13** 2226
[7] Caves C M and Schumaker B L 1985 *Phys. Rev.* A **31** 3068
[8] Radcliffe 1971 *J. Phys. A: Gen. Phys.* **4** 313
[9] Arecchi F T, Courtens E, Gilmore R and Thomas H 1972 *Phys. Rev.* A **6** 2211
[10] Buzek V and Quang T 1989 *J. Opt. Soc. Am.* B **6** 2447
[11] Stoler D, Saleh B E A and Teich M C 1985 *Opt. Acta* **32** 345
[12] Holstein T and Primakoff H 1940 *Phys. Rev.* **58** 1098
[13] Fan H Y and Jing S C 1994 *Phys. Rev.* A **50** 1909
[14] Joshi A and Lawande S V 1989 *Opt. Commun.* **70** 21
[15] Agarwal G S 1992 *Phys. Rev.* A **45** 1787
[16] Gerry C C 1983 *J. Phys. A: Math. Gen.* **16** L1

[17] Fu H C and Sasaki R 1997 *J. Math. Phys.* **38** 3968
[18] Katriel J, Solomon A I, D'Ariano G and Rasetti M 1986 *Phys. Rev.* D **34** 2332
[19] Barut A O and Girardello L 1971 *Commun. Math. Phys.* **21** 41
[20] See, for example, Yurke B and Stoler D 1986 *Phys. Rev. Lett.* **57** 13
[21] Agarwal G S 1988 *J. Opt. Soc. Am.* B **5** 1940
[22] Aragone C, Chalbaud E and Salamo S 1976 *J. Math. Phys.* **17** 1963
[23] Wódkiewicz K and Eberly J H 1985 *J. Opt. Soc. Am.* B **2** 458
[24] Mandel L 1979 *Opt. Lett.* **4** 205
[25] Schleich W and Wheeler J A 1987 *Nature* **326** 574
[26] Loudon R 1980 *Rep. Prog. Phys.* **43** 58
[27] Hillery M 1987 *Phys. Rev.* A **36** 3796
[28] Bergou J A, Hillery M and Yu D 1991 *Phys. Rev.* A **43** 515
[29] Prakash G S and Agarwal G S 1994 *Phys. Rev.* A **50** 4258
[30] Marian P 1997 *Phys. Rev.* A **55** 3051
[31] Hillery M 1989 *Phys. Rev.* A **40** 3147
[32] Gerry C C and Grobe R 1995 *Phys. Rev.* A **51** 4123
[33] Hillery M and Mlodinow 1993 *Phys. Rev.* A **48** 1548
[34] Brif C and Mann A 1996 *Phys. Rev.* A **54** 4505
[35] Gerry C C and Grobe R 1997 *Quantum Semiclass. Opt.* **9** 59
[36] Erdélyi A (ed) 1953 *Bateman Manuscript Project: Higher Transcendental Functions* (New York: McGraw-Hill)
[37] Truax D R 1985 *Phys. Rev.* D **31** 1988
[38] Vidiella-Barranco A and Roversi J A 1994 *Phys. Rev.* A **50** 5233
[39] Walls D F and Milburn G C 1994 *Quantum Optics* (Berlin: Springer)
[40] Wigner E 1932 *Phys. Rev.* **40** 749
[41] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1882
[42] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1857
[43] Vogel K, Akulin V M and Schleich W P 1993 *Phys. Rev. Lett.* **71** 1816
[44] Law C K and Eberly J H 1996 *Phys. Rev. Lett.* **76** 1055
[45] Vidiella-Barranco A and Roversi J A 1998 *Phys. Rev.* A **58** 3349
[46] Dakna M, Clausen J, Knöll L and Welsch D-G 1999 *Phys. Rev.* A **59** 1658